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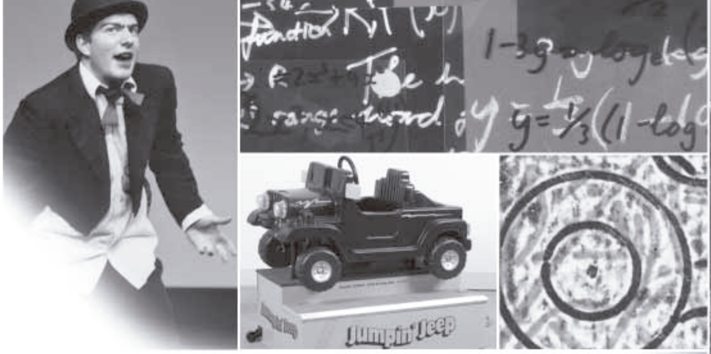


SELECTIONS FROM VCE SEASON OF EXCELLENCE

VCAA Bulletin Supplement 2

VCE Mathematics 2006–2009 Further Mathematics study advice

Data analysis core, Number patterns, Business-related mathematics and Networks and decision mathematics module

The VCE Mathematics study 2006–2009 incorporates various revisions and refinements to the *Data analysis* core and the modules in the *Applications* area of study for Further Mathematics Units 3 and 4. It also includes a new module *Matrices* in the *Applications* area of study, for which advice was published as Supplement 3 to the April 2005 VCAA Bulletin, No. 24. The main changes from the previous study were summarised on page 13 of Supplement 2 to the March 2005 VCAA Bulletin, No. 23.

The following provides further advice on revisions and refinements to the *Data analysis* core and the *Number patterns*, *Business-related mathematics* and *Networks and decision mathematics* modules. This advice should be used in conjunction with that published in the Advice for teachers (Units 3 and 4: Further Mathematics) section of the VCE Mathematics study design 2006–2009.

Core material: Data analysis

Displaying, summarising and describing univariate data

- modelling bell shaped distributions by the normal distribution; the 68-95-99.7% rule and its use in giving meaning to the standard deviation and determining appropriate percentages and intervals, including standard z scores and their application to comparing data values from different distributions.

The inclusion of consideration of z scores is intended to provide students with a tool to make comparisons of spread in data sets that contain data of different orders of magnitude.

For example, the following tables contain the time (in seconds) for a group of 50 students to run (a) 100 m, and (b) 10 000 m.

(a) 100 m times for 50 students (Table 1)

Student	*Time*	*Student*	*Time*	*Student*	*Time*	*Student*	*Time*	*Student*	*Time*
1	12.5	11	12.66	21	12.85	31	12.97	41	13.23
2	12.5	12	12.66	22	12.85	32	12.97	42	13.25
3	12.56	13	12.73	23	12.87	33	12.97	43	13.27
4	12.58	14	12.74	24	12.87	34	12.98	44	13.27
5	12.59	15	12.76	25	12.89	35	12.99	45	13.42
6	12.6	16	12.76	26	12.91	36	13.01	46	13.43
7	12.61	17	12.78	27	12.92	37	13.08	47	13.52
8	12.61	18	12.79	28	12.93	38	13.11	48	13.58
9	12.63	19	12.8	29	12.94	39	13.16	49	14.25
10	12.64	20	12.83	30	12.94	40	13.23	50	14.6

(b) 10 000 m times for 50 students (Table 2)

Student	*Time*	*Student*	*Time*	*Student*	*Time*	*Student*	*Time*	*Student*	*Time*
1	2248	11	3135	21	4383	31	3827	41	4431
2	2818	12	3167	22	2715	32	3897	42	3973
3	2456	13	4383	23	3316	33	3846	43	4429
4	2614	14	1786	24	4383	34	4023	44	4939
5	2941	15	3109	25	2609	35	2838	45	3896
6	2248	16	2884	26	4916	36	3842	46	4916
7	2609	17	3109	27	3701	37	3932	47	6181
8	3109	18	2198	28	4011	38	3853	48	4326
9	3236	19	3163	29	2456	39	3799	49	3809
10	3849	20	4383	30	3300	40	5392	50	5808

Individual pieces of data are compared in terms of both distance from the mean and the variability of the data set.

Student 25 has times of 12.89 seconds for the 100 minutes and 43 minutes 29 seconds (2609 seconds) for the 10 000 m. Both these times are better than the average times for the distances but comparatively, which performance is better?

The relevant data values and statistics required to calculate standardised scores are as follows.

	100 metres	10 000 metres
x	12.89	2609
$\bar{x}$	12.97	3623
s	0.403	953.0

Calculating z scores for each time:

$$Z_{100} = \frac{x-\bar{x}}{s_x} = \frac{12.89-12.97}{0.403} = -0.199$$

$$Z_{10000} = \frac{x-\bar{x}}{s_x} = \frac{2609-3623}{953} = -1.064$$

These z scores imply that the student's time for the 10 000 m is comparatively better. Being more than 1 standard deviation below the mean, this time is in the top 16% of times for the 10 000 m. The student's time for the 100 m is within the middle 68% of times, that is ±1 standard deviation from the mean.

Introduction to regression

- estimation of the equation of an appropriate line of best fit from a scatterplot, use of the formulas $b = r\frac{s_y}{s_x}$ and $a = \bar{y} - b\bar{x}$; and use of technology with bivariate statistics to determine the coefficients of the corresponding equation $y = a + bx$, of the least squares regression line

The relationships $b = r\frac{s_y}{s_x}$ and $a = \bar{y} - b\bar{x}$ provide the connection between the univariate summary statistics (means $\bar{x}$ and $\bar{y}$ and standard deviations, s_x and s_y) for two sets of discrete numerical data, and their corresponding bivariate summary statistics (pearsons correlation coefficient, *r*, coefficients for gradient, *b*, and vertical axis intercept, *a*, of the corresponding linear regression line). For example, consider the following data set of heights and corresponding weights of men for a sample from a population:

Height (*x* cms)	Weight (*y* kgs)
150	48
155	60
162	65
167	63
172	68
179	72
182	89
185	78
192	92
205	100

The univariate summary statistics for the mean and standard deviation for *x* and *y* (obtained using technology) are respectively 174.9, 17.0649087 and 73.5, 16.1675257.

If *x* is used as the *independent* variable (for example to obtain a least squares regression line to identify weight given height), the pearsons correlation coefficient and the equation of the corresponding least squares regression line (obtained using technology) are $r = 0.9567$ and $y = -85.024 + 0.906x$.

The gradient, *b*, of the least squares regression line is related to *r*, s_x and s_y:

$$b = r\frac{s_y}{s_x} = 0.95667625 \times \frac{16.1675257}{17.0649087} = 0.90636804 = 0.906 \text{ correct to 3 decimal places}$$

The related *y* axis intercept is then determined by substitution of the coordinates

$(\bar{x}, \bar{y}) = (174.9, 73.5)$ into the formula: $a = \bar{y} - b\bar{x}$:

$73.5 = a + 0.90636804 \times 174.9 \Rightarrow 73.5 - 0.90636804 \times 174.9 = a$

so $a = -85.024$ correct to 3 decimal places.

While students continue to be expected to use technology to determine *r*, *a* and *b*, for two sets of discrete numerical data, they should also be familiar with the relationship between *b*, s_x and s_y given *r*; and also the relationship between *a*, $\bar{x}$ and $\bar{y}$ given *b* and their *graphical interpretation* with respect to the corresponding scatterplot of the bivariate data and its least squares regression line.

From $b = r\frac{s_y}{s_x}$, students should appreciate that a positive *r* value gives a positive *b* value (a positive gradient), and a negative *r* value will give a negative *b* value. If *b* equals zero, then the equation of the line of best fit simply becomes $y = a$.

Additional examples of linear regression analysis can be found at the following website.
www.stat.yale.edu/Courses/1997-98/101/linreg.htm
Overview of linear regression procedures

http://people.hofstra.edu/faculty/Stefan_Waner/RealWorld/calctopic1/regression.html – Waner and Costenoble
Regression techniques and analysis

Displaying, summarising and describing time series data

- Forecasting using trend lines (with the data deseasonalised where necessary).

The addition of forecasting to the *Time series* topic provides a purpose and use for trend lines. That is, the analysis of time series data is useful for not only observing trends in the past, but also to enable predictions of data values in the future. This has obvious value, for example to economists and social planners.

Only trend line forecasting is required. Forecasting using moving averages or exponential smoothing is **not** included in the course. Trend lines should be developed from deseasonalised data where applicable; in which case it can be expected that students will seasonalise forecasts to get predicted values.

Students should also appreciate that the reliability of forecasted data will lessen as the time to the predicted value increases.
The following website provides an overview of forecasting techniques.
http://people.brunel.ac.uk/~mastjjb/jeb/or/forecast.html – J. E. Beasley
Summary of forecasting approaches.

Module material: Applications

Module 1: Number patterns

- fibonacci and related sequences and applications (numerical and graphical solution of related equations)

The *fibonacci* sequence is an example of a simple linear second order difference equation (recurrence relation), characterised by the recursion relation that any term is the *sum* of the preceding *two* terms. It is called *the* fibonacci sequence as it was studied by the merchant Leonardo of Pisa, also called Fibonacci (meaning 'son of Bonaccio') in his book the *Liber Abaci* (1202). Fibonacci developed the sequence as a solution to a population modelling problem about pairs of breeding rabbits. The problem can be stated as follows:

'Begin with a single pair of fertile breeding rabbits, how many pairs of rabbits will be produced in a year if: each pair gives birth to a new pair of rabbits each month; a new pair become fertile from the second month; fertile pairs are always productive; and there are no deaths?'

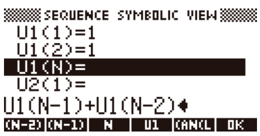
The fibonacci sequence is generally taken to be $F = \{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89 \ldots\}$, which describes the number of mature (breeding) pairs of rabbits in the n th month. The terms of this sequence are called the *fibonacci* numbers. Closely related sequences are $\{0, 1, 1, 2, 3, 5 \ldots\}$ which describes the number of new-born (fertile the *following* month) pairs of rabbits in the n th month; and $\{1, 2, 3, 5, 8, 13 \ldots\}$ which describes the total number of pairs of rabbits (mature *and* new born) in the n th month. *The* fibonacci sequence is thus generally specified by the linear second order difference equation (recurrence relation):

$F(1) = 1, F(2) = 1$ and $F(n + 2) = F(n + 1) + F(n)$ for $n = 1, 2, 3 \ldots$

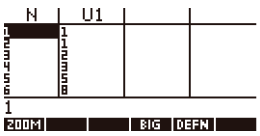
Some references use the alternative form of specification:

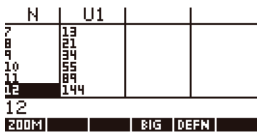
$F(0) = 1, F(1) = 1$ and $F(n) = F(n - 1) + F(n - 2)$ for $n = 2, 4, 5 \ldots$

The following 'screen-dumps' illustrate the sub-sequence of the first few terms of F obtained from a graphics calculator or CAS:



which gives the sequence values





There are many contexts for which fibonacci and related sequences can be used as models in nature, information technology, mathematics, art and design, and discussion of a range of these, with diagrams and photographs, can be found at the websites referenced below. One or more of these contexts could be used to develop a suitable analysis task for School-assessed Coursework.

There *is* an explicit formula, or rule, for generating terms of the fibonacci sequence:

$$F(n) = \frac{\varphi^n}{\sqrt{5}} - \frac{(1-\varphi)^n}{\sqrt{5}} = \frac{\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n}{\sqrt{5} \times 2^n}$$

where $\varphi = \dfrac{1+\sqrt{5}}{2}$ is the *golden ratio*. Students are **not** expected to know this formula; however, exploration of the relationship between the fibonacci sequence and the golden ratio, with or without consideration of this formula, could be used as the basis for developing an analysis task for school assessed coursework. This could include, for example, comparison of growth for a fibonacci sequence, with exponential or other growth sequences.

It is of note that the above formula uses a *surd* term, $\sqrt{5}$, in the calculation of a sequence of *whole number* values. An alternative approach, which is **not** required as part of the module content, uses 2×2 matrices to calculate a sub-sequence of three consecutive terms of the fibonacci sequence.

If $F = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ then, for a given value of n, the matrix $F^n = \begin{bmatrix} F(n+1) & F(n) \\ F(n) & F(n-1) \end{bmatrix}$ calculates the sub-sequence of the fibonacci sequence $\{F(n-1), F(n), F(n + 1)\}$.

This approach (which has the advantage over the explicit formula of calculating whole number values using whole numbers) could also be used as the basis of an analysis task for school assessed coursework.

More generally, if the *initial* terms of a fibonacci sequence are taken to be arbitrary integers a and b, then a *related* sequence f is generated, specified by:

$f(1) = a \quad f(2) = b$ and $\quad f(n + 2) = f(n + 1) + f(n)$ for $n = 1, 2, 3 \ldots$

For example, if $a = 2$ and $b = 4$, then the corresponding related sequence S is:

$S = \{2, 4, 6, 10, 16 \ldots\}$.

These related sequences can also be expressed in terms of *the* fibonacci sequence by the relation:

$f(n + 2) = bF(n + 1) + aF(n)$ for $n = 1, 2, 3 \ldots$

Thus, for the sequence S, when $n = 3$, then:

$s(5) = 4 \times F(4) + 2 \times F(3) = 4 \times 3 + 2 \times 2 = 16$

A particular related sequence, where $a = 1$ and $b = 3$ is the *lucas* sequence $L = \{1, 3, 4, 7, 11, 18 \ldots\}$ and its terms are called the lucas numbers. Edouard Lucas was a nineteenth-century French mathematician who studied recursion and sequences, and gave the fibonacci sequence its name. The lucas and fibonacci numbers are linked by the relation $L(n) = F(n + 1) + F(n - 1)$. For example,

$L(6) = 18 = F(7) + F(5) = 13 + 5$

Students should be able to work with fibonacci and related sequences from first principles, using numerical and graphical approaches, with the assistance of technology as applicable. They should be able to identify a sequence as being a fibonacci or related sequence or not; generate such sequences, or sub-sequences of them, by hand or using technology; represent these using lists, tables or graphs; and solve related problems graphically and/or numerically.

Such work would be based on knowledge of the difference equation (recurrence relation) $f(n + 2) = f(n + 1) + f(n)$ and identification of relevant information with respect to terms of the sequence, perhaps from a situation under consideration, that enables a specific fibonacci or related sequence to be determined.

The following websites provide a practical approach to fibonacci and related sequences. They discuss a range of situations where such sequences arise and their applications.

www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/fib.html – Ron Knott, Surrey University

A very accessible general discussion of fibonacci numbers, the golden section (ratio) and applications in art, architecture and design.

www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/fibnat.html – Ron Knott, Surrey University

Provides a detailed range of example of Fibonacci and related sequences in nature, with an extensive set of diagrams and photographs.

www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Fibonacci.html – School of Mathematics and Statistics, University of St Andrews

Provides historical and mathematical background on the work of Fibonacci, and related mathematics topics, history and mathematicians.

The following websites provide more detailed mathematical background on fibonacci and related sequences.

http://theory.cs.uvic.ca/~cos/amof/e_fiboI.htm – Canada's SchoolNet

Covers a range of standard and some less common applications of Fibonacci sequences, along with links to other relevant sites and sources of information.

http://en.wikipedia.org/wiki/Fibonacci_number – Wikimedia

A thorough and accessible discussion of the mathematical theory behind fibonacci and related sequences.

http://mathworld.wolfram.com/FibonacciNumber.html – Wolfram Research

A comprehensive reference with detailed discussion of the mathematics of Fibonacci and related sequences. Contains a detailed bibliography for text references.

Module 2: Business related mathematics

The content of this module is largely unchanged from the current study. It has, however, been reconfigured to more accurately reflect current loan and investment conditions and the use of supporting technology to readily evaluate and analyse them.

The determination of percentage change in relation to fees and charges, discounting, mark-ups, and other financial applications has been made explicit. The ordinary perpetuity as a special form of an investment annuity has been a valid component of the current study, but has now been clearly identified in the new study.

Time Value of Money (TVM) applets, preloaded into many graphic calculators or available as auxiliary programs, can be used in many applications in this module as illustrated in the following example.

Lou is 25 years of age and has just started a new job. Her yearly salary is $42 000. Her employer contributes each month 9% of her gross salary into a superannuation fund and Lou also contributes 5%. The fund advertises a nominal return of 8% pa (compounding monthly).

Based on these figures, what is the amount of superannuation Lou will have available at age 60?

Using a TVM will give the solution:

N = 420	I% = 8
PV = 0	PMT = -490
FV = 1124002.42	
P/Y = 12	C/Y = 12
PMT: END	

Lou is very pleased with this potential outcome, particularly when she considers that over the next 35 years her wage, and therefore superannuation contributions will regularly rise, leading to an even larger final figure.

This amount however will not have the same buying power in 35 years that it does today due to the effects of inflation. Using a figure of 5% pa loss in value each year, determine the purchasing power of \$1 124 002.42 in 35 years time.

Purchasing Value in 35 years = 0.95^{35}(1 124 002.42)

= \$186 678.13

Lou's superannuation payout will be placed in a perpetuity which will provide a monthly income without using any of the principal. With forecast investment interest rates of 6% pa (compounding monthly), what income can Lou anticipate in her retirement?

The payment in perpetuity equals the interest earned. It is an application of simple interest,

$$\text{Payment} = \frac{PRT}{100} = \frac{1124002.42 \times 6}{100 \times 12} = \$5620.01 \text{ per month.}$$

(P = principal, R = annual interest rate, T = years)

or it can be solved using the TVM using any value for N,

N = 100	I = 6%
PV = -1124002.42	
PMT = 5620.01	
FV = 1124002.42	P/Y = 12
C/Y = 12	PMT: END

If Lou was prepared to draw all of her funds over a 30 year period, what would her monthly income be? Assume investment interest rates remain at 6% paid monthly.

Using a TVM,

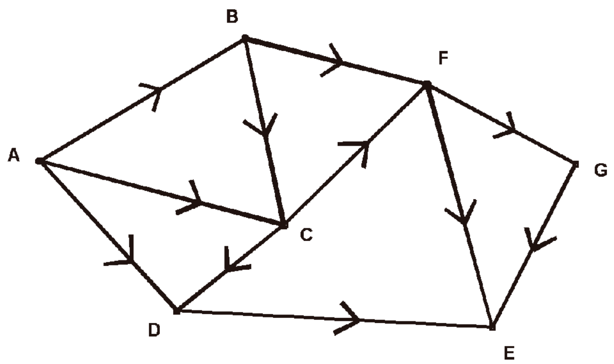
N = 360	I = 6%
PV = -1124002.42	
PMT = 6738.96	
FV = 0	P/Y = 12
C/Y = 12	PMT: END

Module 5: Networks and decision mathematics

- graphical and matrix representation of directed graphs (networks), including application to dominance and reachability
- critical path analysis, including network construction (with activity as edge), location of the critical path by forward and backward scanning
- assignment problems, for example representation by bipartite graph; optimal allocation including use of the hungarian algorithm

The changes in this module are minor with the dominance and reachability applications being the most significant. Defining critical path construction using the activity as edge method (also referred to as Activity on Arrow) specifies the approach that has been used in setting Further Mathematics examination questions. Teachers should note that the commonly used software *Microsoft Project* uses an alternative procedure with the activities represented by the nodes. The reaccredited study specifies knowledge and use of the hungarian algorithm rather than have it as a suggested approach.

The *reachability* of a network refers to the existence, and number of, pathways between vertices in a directed graph.



In the above network, there are two pathways from A to C – one of one-edge in length (A→C) and one of two-edges in length (A→B→C). Related problems in Further Mathematics can be solved by inspection, although the adjacency matrix can also be used. This could be the basis for a suitable analysis task as part of school based assessment. For the above network the adjacency network, **M** (with rows and columns in alphabetical order A to G) is

$$\mathbf{M} = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

The adjacency matrix shows the number of one-edge pathways between vertices. Successive powers of the adjacency matrix will show the number of 2, 3, 4 etc. pathways between two vertices

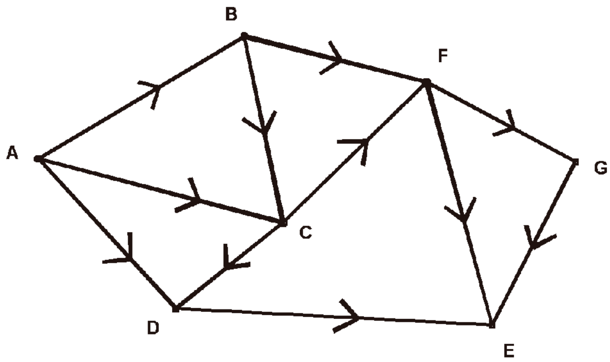
$$\mathbf{M}^2 = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Reading from $\mathbf{M}^2$, there are 2 two-edge pathways from A to F (A→B→F and A→C→F) and 2 two-edge pathways from C to E (C→F→E and C→D→E)

$$\mathbf{M}^4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 4 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Reading from $\mathbf{M}^4$ there are 4 four-edge pathways from A to E (A→B→F→G→E, A→B→C→D→E, A→B→C→F→E and A→C→F→G→E)

Dominance is determined in a similar way to reachability. Consider the same network as used above as showing the outcomes of games between 7 teams, A to G. Each directed edge represents a win by one team over another.



The simplest consideration of dominance can be determined by adding the row elements of the adjacency matrix to give dominance values of

Team	Dominance value
A	3
B	2
C	2
D	1
E	0
F	2
G	1

From this table, A is the most dominant team, teams B, C and F equal second, teams D and G equal fifth, and team E the least dominant.

The determination of dominance can be extended with consideration of indirect dominance between teams. For example Team C beat team D which in turn has beaten team E. Although team C has not played team E, we can say that C is dominant over E from the (two-edge) pathway through D.

As shown above we can find the number of two-edge pathways by squaring the adjacency matrix.

Therefore a more extensive consideration of dominance can be found using $\mathbf{M} + \mathbf{M}^2$.

$$\mathbf{M} + \mathbf{M}^2 = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 & 2 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

This gives row totals of

Team	Dominance value
A	8
B	6
C	5
D	1
E	0
F	3
G	1

This gives a dominance order of A, B, C, F, D and G, and E.

The use of $\mathbf{M} + \mathbf{M}^2$ gives equal weighting to direct wins over teams and indirect 'wins' through other teams. To give less weighting to the indirect wins, fractional values of $\mathbf{M}^2$ can be used, such as $\mathbf{M} + 0.5\mathbf{M}^2$. In addition, consideration of indirect wins through two other teams can be included with use of $\mathbf{M} + \mathbf{M}^2 + \mathbf{M}^3$.

Matrix arithmetic is **not** required for the *Networks and decision mathematics* module. This effectively limits the investigation of dominance relationships to situations that can be tackled by inspection.

The following websites provide a range of background information and examples.

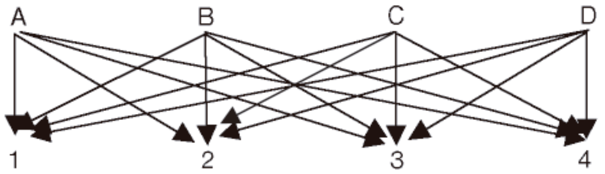
www.cs.usask.ca/resources/tutorials/csconcepts/1999_8/tutorial/index.html
Basic graph theory and common algorithms.

www.math.ucdavis.edu/~daddel/linear_algebra_appl/Applications/GraphTheory/GraphTheory_9_17/node11.html
– Ali A. Daddel
Diagramatic display of dominance principles.

http://aix1.uottawa.ca/~jkhoury/graph.htm
Dominance principles and also contains some basic graph theory.

http://mathworld.wolfram.com/Tournament.html – Wolfram Research
Discusses characteristics of tournaments in which all players play each other.

The hungarian algorithm is used to find optimal solutions in allocation problems characterised by the use of bipartite graphs. For example, four workers A, B, C, D can each complete four tasks 1, 2, 3, 4.



The time (hours) taken by each worker to complete each task is shown in the matrix:

$$\begin{array}{cc} & \text{Tasks} \\ & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \text{Workers} \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 8 & 6 & 5 & 3 \\ 5 & 2 & 6 & 7 \\ 2 & 3 & 1 & 5 \\ 5 & 7 & 6 & 3 \end{bmatrix} \end{array}$$

Application of the hungarian algorithm on the matrix is as follows:

1. For each row subtract the smallest row element from all elements in that row (begin with row reduction unless the question states otherwise). For example, 3 is subtracted from all elements in row 1. This ensures that every row has at least 1 zero element.

$$\begin{bmatrix} 5 & 3 & 2 & 0 \\ 3 & 0 & 4 & 5 \\ 1 & 2 & 0 & 4 \\ 2 & 4 & 3 & 0 \end{bmatrix}$$

2. For each column subtract the smallest column element from all elements in that column. All columns now have at least 1 zero element.

$$\begin{bmatrix} 4 & 3 & 2 & 0 \\ 2 & 0 & 4 & 5 \\ 0 & 2 & 0 & 4 \\ 1 & 4 & 3 & 0 \end{bmatrix}$$

At this stage a unique solution will be possible if each row and each column has exactly one zero each. Allocation can then be made and minimum time determined. That is not the case here.

3. Draw lines through the rows and columns to cover all zeros. The minimum number of lines must be used. In this case only 3 lines are required.

$$\begin{bmatrix} 4 & 3 & 2 & 0 \\ 2 & 0 & 4 & 5 \\ 0 & 2 & 0 & 4 \\ 1 & 4 & 3 & 0 \end{bmatrix}$$

4. *Subtract* the smallest uncovered value from all *uncovered* elements and *add* it to all elements at the *intersection* of lines. The smallest uncovered value is 1.
5. The allocation can now be made by starting at the zeros that do not provide alternative selections. An appropriate order is B→2, A→4, C→3, D→1.

$$\begin{array}{cc} & \text{Tasks} \\ & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \text{Workers} \begin{array}{c} \text{A} \\ \text{B} \\ \text{C} \\ \text{D} \end{array} & \begin{bmatrix} 3 & 2 & 1 & 0 \\ 2 & 0 & 4 & 6 \\ 0 & 2 & 0 & 5 \\ 0 & 3 & 2 & 0 \end{bmatrix} \end{array}$$

6. The minimum total time for completion of all tasks, using the times shown in the original matrix is
$2 + 3 + 1 + 5 = 11$ hours.

Other examples of use of the Hungarian algorithm can be found at the following websites.

www.mcs.vuw.ac.nz/courses/MATH214/2005T1/matching_3.pdf – Victoria University of Wellington
Discusses the efficacy of the Hungarian algorithm in finding an optimum solution.

www.blacksacademy.co.uk/data/MARFHA11.pdf – Black's academy
Provides a variation in description of the algorithm.
All websites accessed 26 July 2005.

Notes

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